Inverse Laplace Transform

If the Laplace transform of a function $F(t)$ is $f(s)$ i.e. $L\{F(t)\} = f(s)$; Then $F(t)$ is called the inverse Laplace transform of $f(s)$ i.e.

$$F(t) = L^{-1}\{f(s)\}$$

where, L^{-1} is called inverse Laplace transform operator.

Important Formulae

1. $L^{-1}\left(\frac{1}{s}\right) = 1$
2. $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$
3. $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
4. $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$
5. $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$
6. $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$
7. $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
8. $L^{-1}[F(s-a)] = e^{at} f(t)$
9. $L^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = \frac{1}{b}e^{at}\sin bt$
10. $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$
11. $L^{-1}\left[\frac{b}{(s-a)^2-b^2}\right] = \frac{1}{b}e^{at}\sinh bt$
12. $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt$
13. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$
14. $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}t\sin at$
15. $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$
16. $L^{-1}(1) = \delta(t)$
17. $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}[\sin at + at\cos at]$
18. $L^{-1}\left\{\frac{1}{s}f(s)\right\} = \int_0^t f(t)\,dt$

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Ex-1:- $\frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$

Soln:- $L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{(n-1)!} = \frac{t^{n-1}}{\sqrt{n}}$

$= \frac{t^{1/2-1}}{\sqrt{1/2}} = \frac{t^{-1/2}}{\sqrt{1/2}} = \frac{t^{-1/2}}{\sqrt{\pi}}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^{1/2}}\right\} = \frac{1}{\sqrt{\pi t}} \Rightarrow \frac{1}{s^{1/2}} = L\left[\frac{1}{\sqrt{\pi t}}\right]$

Ex-2:- $L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$ $[a=2]$

i) $L^{-1}\left[\frac{1}{s^2-9}\right] = L^{-1}\left[\frac{1}{s^2-3^2}\right] = L^{-1}\left[\frac{1}{3} \times \frac{3}{s^2-3^2}\right] = \frac{1}{3}\sin 3t$

ii) $L^{-1}\left[\frac{s}{s^2+9}\right] = L^{-1}\left[\frac{s}{s^2+3^2}\right] = \cos 3t$

iv) $L^{-1}\left[\frac{1}{s^2+36}\right] = L^{-1}\left[\frac{1}{s^2+6^2}\right] = L^{-1}\left[\frac{1}{6} \cdot \frac{6}{s^2+6^2}\right] = \frac{1}{6}\sin 6t$

v) $L^{-1}\left[\frac{1}{(s+3)^2-4}\right] = L^{-1}\left[\frac{1}{2} \frac{2}{\{s-(-3)\}^2-2^2}\right] = \frac{1}{2} e^{-3t}\sin 2t$

vi) $L^{-1}\left[\frac{s+2}{(s+2)^2-25}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2-5^2}\right] = e^{2t}\cos 5t$

vii) $L^{-1}\left[\frac{1}{2s^5}\right] = \left[\frac{1}{2} \times \frac{t^{5-1}}{(5-1)!}\right] = \frac{1}{2}\frac{t^4}{4!} = \frac{t^4}{48}$

Q. $L^{-1}\left[\frac{s^2+2s+6}{s^3}\right] = L^{-1}\left[\frac{1}{s} + 2\frac{1}{s^2} + \frac{6}{s^3}\right]$

$= 1 + 2\frac{t^{2-1}}{(2-1)!} + 6\frac{t^{3-1}}{(3-1)!} \Rightarrow 1 + \frac{2t}{1!} + \frac{6t^2}{2!} = 1 + 2t + 3t^2$

Q. $L^{-1}\left[\frac{2s-5}{9s^2-25}\right] = L^{-1}\left[\frac{2s}{9s^2-25} - \frac{5}{9s^2-25}\right]$

$= L^{-1}\left[\frac{2/9\, s}{s^2-25/9} - \frac{5/9}{s^2-25/9}\right]$

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$$= L^{-1}\left[\frac{2}{9} \frac{s}{s^2-(5/3)^2} - \frac{5/9}{s^2-(5/3)^2} \right] \qquad 5/9 \times 3 \times \frac{1}{8} = 5/3$$

$$= L^{-1}\left[\frac{2}{9} \frac{s}{s^2-(5/3)^2} - \frac{1}{3} \frac{5/3}{s^2-(5/3)^2} \right]$$

$$= \frac{2}{9} \cos h\, 5/3\, t - \frac{1}{3} \sin h\, 5/3\, t$$

Q. $L^{-1}\left[\frac{s-2}{6s^2+20} \right] = L^{-1}\left[\frac{s}{6s^2+20} - \frac{2}{6s^2+20} \right]$

$$= L^{-1}\left[\frac{5/6}{s^2+20/6} - \frac{2/6}{s^2+20/6} \right]$$

$$= L^{-1}\left[\frac{1}{6} \frac{s}{s^2+(\sqrt{10/3})^2} - \frac{1/3}{s^2+(\sqrt{10/3})^2} \right]$$

$$= L^{-1}\left[\frac{1}{6} \frac{s}{s^2+(\sqrt{10/3})^2} - \frac{1}{\sqrt{30}} \frac{\sqrt{10/3}}{s^2+(\sqrt{10/3})^2} \right]$$

$$= \frac{1}{6} \cos \sqrt{10/3}\, t - \frac{1}{\sqrt{30}} \sin \sqrt{10/3}\, t$$

Q. $L^{-1}\left[\frac{6}{2s-3} - \frac{3+4s}{9s^2-16} \right]$

$$= L^{-1}\left[\frac{6/2}{s-3/2} - \frac{3}{9s^2-16} - \frac{4s}{9s^2-16} \right]$$

$$= L^{-1}\left[\frac{3}{s-3/2} - \frac{1/3}{s^2-16/9} - \frac{4/9\, s}{s^2-16/9} \right]$$

$$= L^{-1}\left[3 \cdot \frac{1}{s-3/2} - \frac{1/3}{s^2-(4/3)^2} - \frac{4/9\, s}{s^2-(4/3)^2} \right]$$

$$= L^{-1}\left[3 \cdot \frac{1}{s-3/2} - \frac{1}{4} \frac{4/3}{s^2-(4/3)^2} - 4/9 \frac{s}{s^2-(4/3)^2} \right]$$

$$= 3 e^{3/2 t} - \frac{1}{4} \sin h\, 4/3\, t - 4/9 \cos h\, 4/3\, t$$

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Q $L^{-1}\left[\frac{3s-8}{4s^2+25}\right] = \frac{3}{4}\cos 5t/2 - \frac{4}{5}\sin 5t/2$

Q. $L^{-1}\left[\frac{3(s^2-2)^2}{2s^5}\right] = 3/2 - 3t^2 + 1/2 t^4$

Q $L^{-1}\left[\frac{2s-s^2}{4s^2+25} + \frac{4s-18}{9-s^2}\right] = \frac{1}{2}\left(\cos 5t/2 - \sin 5t/2\right) - 4\cosh 3t + 6\sinh 3t$

Q. $L^{-1}\left[\frac{1}{4s} + \frac{16}{1-s^2}\right] = \frac{1}{4} - 16\sinh t$

Multiplication by s

$$L^{-1}[sF(s)] = \frac{d}{dt} f(t) + f(0)\delta(t)$$

2. $F(s) = \frac{s}{s^2+1}$

$L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$L^{-1}\left[\frac{s}{s^2+1}\right] = \frac{d}{dt}(\sin t) + \sin(0)\delta(t) = \cos t$

Q. $F(s) = \frac{s}{4s^2-25}$

$L^{-1}\left[\frac{1}{4s^2-25}\right] = \frac{1}{4}\left[L^{-1}\frac{1}{s^2-25/4}\right] = \frac{1}{4}\cdot\frac{2}{5}\sinh 5/2\, t$

$= \frac{1}{10}\sinh 5/2\, t$

$L^{-1}\left[\frac{4s}{4s^2-25}\right] = \frac{1}{10}\frac{d}{dt}\sinh 5/2\, t + 0 = 1/10 \times 5/2 \sinh 5/2\, t = \frac{1}{4}\cosh 5/2\, t$

Q. $F(s) = 6\,\frac{3}{2s+9}$

$L^{-1}\left[\frac{3}{2s+9}\right] = 3/2\, L^{-1}\left[\frac{1}{2s+9/2}\right] = 3/2\, e^{-9/2 t}$

$L^{-1}\left[\frac{3s}{2s+9}\right] = -3/2 \frac{d}{dt} e^{-9/2 t} + 3/2\, e^{-9/2(0)}\delta(t)$

$= 3/2 \times -9/2\, e^{-9/2 t} + 3/2 = -\frac{27}{4} e^{-9/2 t} + 3/2$

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Q. $\frac{s}{s+5} = F(s)$

Ans) $L^{-1}\left[\frac{s}{s+5}\right] = \frac{d}{dt} e^{-5t} + e^{-5\cdot 0}\,\delta(t)$

$= -5e^{-5t} + 1$

Q. $F(s) = \frac{2s}{3s+6}$

Ans) $L^{-1}\left[\frac{2s}{3s+6}\right] = \frac{2}{3} \times L^{-1}\left[\frac{s}{s+2}\right]$

$= \frac{2}{3} \times \left[\frac{d}{dt} e^{-2t} + e^{-2(0)}\,\delta(t)\right]$

$= \frac{2}{3} \times t2 e^{-2t} = -4/3\, e^{-2t}$

Q. $F(s) = \frac{s}{2s^2-1}$

$L^{-1}\left[\frac{1}{2s^2-1}\right] = \frac{1}{2} L^{-1}\left[\frac{1}{s^2-(1/\sqrt{2})^2}\right] = \frac{1}{2}$ ~~$\sinh t/\sqrt{2}$~~ $\times 1/\sqrt{2}$

$= \frac{1}{2} \times \frac{1}{\sqrt{2}} \sinh \sqrt{1/2}\, t$

$L^{-1}\left[\frac{s}{2s^2-1}\right] = \frac{1}{2\sqrt{2}} \times \sqrt{\frac{1}{2}} \cosh \sqrt{\frac{1}{2}}\, t$

$= \frac{1}{8} \cosh \frac{1}{\sqrt{2}} t$

Q. $F(s) = \frac{s^2}{s^2+a^2}$

$L^{-1}[F(s)] = L^{-1}\left[\frac{s^2}{s^2+a^2}\right] = \frac{d}{dt} \cos at + \cos a\cdot 0\cdot \delta(t)$

$= -a \sin at + 1$

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Divison by s (Multiplication by $\frac{1}{s}$)

$$L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t \left[L^{-1}[F(s)]\right] dt = \int_0^t f(t)\,dt$$

Q. $\frac{1}{s(s+a)} = F(s)$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left(\frac{1}{s+a}\right) dt$$

$$= \int_0^t e^{-at}\,dt = \frac{e^{-at}}{a}\Big|_0^t = \frac{e^{-at}}{-a} + \frac{1}{a}$$

Q. $L^{-1}\left[\frac{1}{(s^2+1)s}\right] = F(s)$

$$L^{-1}\left[\frac{1}{s(s^2+1)}\right] = \int_0^t L^{-1}\left(\frac{1}{s^2+1}\right) dt$$

$$= \int_0^t \sin t\,dt = [-\cos t]_0^t = -\cos t + 1$$

Q. $F(s) = \frac{s^2+3}{s(s^2+9)} = L^{-1}\left[\frac{s^2+9-6}{s(s^2+9)}\right] = L^{-1}\left[\frac{1}{s} - \frac{6}{s(s^2+9)}\right]$

$$= 1 - 2\int_0^t \sin 3t\,dt$$

$$= 1 + 2 \times \frac{1}{3}[\cos 3t]_0^t = 1 + \frac{2}{3}\cos 3t - \frac{2}{3}$$

$$= \frac{2}{3}\cos 3t + \frac{1}{3} = \frac{1}{3}[2\cos 3t + 1]$$

Q. $F(s) = \frac{s^2+2}{s(s^2+4)}$

$$L^{-1}\left[\frac{s^2+2}{s(s^2+4)}\right] = L^{-1}\left[\frac{s^2+4-2}{s(s^2+4)}\right]$$

$$= L^{-1}\left[\frac{1}{s} - \frac{2}{s(s^2+4)}\right] = 1 - 2 \times \frac{1}{2}\int_0^t \sin 2t\,dt$$

$$= 1 + [\cos 2t]_0^t$$

$$= 1 + \frac{\cos 2t}{2} - 1 = \frac{\cos 2t}{2}$$

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$F(s) = \frac{1}{2s(s-3)}$

$$L^{-1}\left[\frac{1}{2s(s-3)}\right] = \frac{1}{2}\int_0^t L^{-1}\left[\frac{1}{s-3}\right] dt$$

$$= \frac{1}{2}\int_0^t e^{3t}\,dt = \frac{1}{2} \times \frac{e^{3t}}{3}\Big|_0^t = \frac{1}{2}\left[\frac{e^{3t}-1}{3}\right]$$

Q. $F(s) = \frac{1}{s(s^2-16)}$

$$L^{-1}\left[\frac{1}{s(s^2-16)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2-16}\right] dt = \frac{1}{4}\int_0^t \sinh 4t\,dt$$

$$= \frac{1}{4}\left[\frac{-\cosh 4t}{4}\right]_0^t$$

$$= \frac{1}{16}\left[1 - \cosh 4t\right]$$

Q. $F(s) = \frac{1}{s(s^2+a^2)}$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+a^2}\right] dt$$

$$= \frac{1}{a}\int_0^t \sin at\,dt = \frac{1}{a}\left[\frac{-\cos at}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left[1 - \cos at\right]$$

Q. $\frac{1}{s^2(s+1)} = F(s)$

$$L^{-1}\left[\frac{1}{s^2(s+1)}\right] = \int_0^t \int_0^t L^{-1}\left[\frac{1}{s+1}\right] dt$$

$$= \int_0^t \int_0^t e^{-t}\,dt$$

$$= \int_0^t \left[-e^{-t}\right]_0^t dt$$

$$= -\int_0^t (e^{-t} - 1)\,dt = -\int_0^t e^{-t}\,dt + \int_0^t 1\,dt$$

$$= +\left[e^{-t}\right]_0^t + t = e^{-t} - 1 + t$$

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Inverse Laplace transforms of derivatives

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = -t\,L^{-1}[F(s)] = -t f(t)$$

$$\Rightarrow L^{-1}[F(s)] = -\frac{1}{t}\,L^{-1}\frac{d}{ds}F(s)$$

Q. Find $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$.

Sol: $L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\frac{s+1}{s-1}\right]$

$$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s+1) - \frac{d}{ds}\log(s-1)\right]$$

$$= -\frac{1}{t}\left\{L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]\right.$$

$$= -\frac{1}{t}[e^{-t} - e^{t}] = \frac{1}{t}[e^{t} - e^{-t}]$$

Q. Find $L^{-1}\left\{\log\frac{s^2-1}{s^2}\right\}$.

Sol: $L^{-1}\left\{\log\frac{s^2-1}{s^2}\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log(s^2-1) - \frac{d}{ds}\log s^2\right]$

$$= -\frac{1}{t}\left[\cdot = -\frac{1}{t}L^{-1}\left[\frac{1}{s^2-1}\cdot 2s - \frac{d}{ds}2\log s\right]\right.$$

$$= -\frac{1}{t}L^{-1}\left[\frac{2s}{s^2-1} - \frac{2}{s}\right]$$

$$= -\frac{1}{t}\left[2\cosh t - 2\right] = \frac{2}{t}[1 - \cosh t]$$

Q. Find $L^{-1}\left[\tan^{-1}\frac{1}{s}\right]$

$$L^{-1}\left[\tan^{-1}\frac{1}{s}\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\tan^{-1}\frac{1}{s}\right]$$

$$= -\frac{1}{t}L^{-1}\left[\frac{1}{1+\frac{1}{s^2}}\left(-\frac{1}{s^2}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{1}{1+s^2}\right] = \frac{1}{t}\frac{\sin t}{1}$$

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1. $F(s) = \cot^{-1} s/2$

$$L^{-1}\left[\cot^{-1} s/2\right] = 0 - \frac{1}{t} L^{-1}\left[\frac{d}{ds} \cot^{-1} s/2\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{-1}{1+s^2/4} \cdot \frac{1}{2}\right]$$

$$= \frac{1}{t} L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$f(t) = \frac{1}{t} \sin 2t$$

Q. $F(s) = \tan^{-1}(1+s)$

$$L^{-1}\left[\tan^{-1}(1+s)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds} \tan^{-1}(1+s)\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{1+(1+s)^2}\right]$$

$$= -\frac{1}{t} L^{-1}\left[\frac{1}{(s+1)^2+1}\right] \qquad a=-1,\ b=1$$

$$= -\frac{1}{t} e^{-t} \sin t$$

Q. $F(s) = \frac{2as}{(s^2+a^2)^2}$

$L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right]$ · $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$

$$L^{-1}\left[\frac{d}{ds}\left\{\frac{a}{s^2+a^2}\right\}\right] = -t \sin at$$

$$\Rightarrow L^{-1}\left[\frac{-2as}{(s^2+a^2)^2}\right] = -t \sin at$$

$$\Rightarrow L^{-1}\left[\frac{2as}{(s^2+a^2)^2}\right] = t \sin at$$

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Q. $F(s) = \frac{s^2 - a^2}{(s^2+a^2)^2}$

Ans) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \cos at$

$\Rightarrow L^{-1}\left[\frac{d}{ds}\frac{s}{(s^2+a^2)}\right] = -t\cos at$

$\Rightarrow L^{-1}\left[\frac{1\cdot s^2+a^2 - s(2s)}{(s^2+a^2)^2}\right] = -t\cos at$

$\Rightarrow L^{-1}\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right] = -t\cos at$

$\Rightarrow L^{-1}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = -t\cos at$

$\Rightarrow -L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = -t\cos at$

$\Rightarrow L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$

Q. $F(s) = \log\left(1+\frac{\omega^2}{s^2}\right)$

$L^{-1}\left[\log\left(1+\frac{\omega^2}{s^2}\right)\right] = -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{\omega^2}{s^2}\right)\right]$

$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+\omega^2}{s^2}\right)\right]$

$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\frac{s^2}{(s^2+\omega^2)}\right]$

$= -\frac{1}{t}L^{-1}\left[\frac{d}{ds}\log s^2+\omega^2 - 2\frac{d}{ds}\log s\right]$

$= -\frac{1}{t}L^{-1}\left[\frac{1}{s^2+\omega^2}\cdot 2s - 2\frac{1}{s}\right]$

$= -\frac{1}{t}\left[2\cos\omega t - 2\right] = \frac{2}{t}\left[1-\cos\omega t\right]$

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Q. $\log(1+1/s) \rightarrow \frac{1}{t}(1-e^{-t})$

Q. $\frac{s}{(s^2+a^2)^2} \rightarrow \frac{t \sin at}{2a}$

Q. $\log\left(1+\frac{1}{s^2}\right] \rightarrow 2/t(1-\cos \omega t)$

Q. $\log\left\{\frac{s^2+b^2}{(s-a)^2}\right\} \rightarrow \frac{e^{-at} - \cos bt}{t}$

Inverse Laplace Transform of Integrals

$$L^{-1}\left[\int_s^\infty F(s)\,ds\right] = \frac{f(t)}{t} = \frac{1}{t} L^{-1}[F(s)]$$

$$\Rightarrow L^{-1} F(s) = t\, L^{-1}\left[\int_s^\infty F(s)\,ds\right]$$

Ex:- $L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = t\, L^{-1} \int_s^\infty \frac{2s\,ds}{(s^2+1)^2}$

$$= t\, L^{-1}\left[-\frac{1}{s^2+1}\right]_s^\infty$$

$$= t\, L^{-1}\left[\frac{1}{\infty} + \frac{1}{s^2+1}\right]$$

$$= t\, L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= t \sin t.$$

Partial Fraction method:- It is re-expressing a rational function (a ratio of polynomial functions) as a sum of simpler fractions.

Q. $\frac{1}{s^2-5s+6} = F(s)$

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = F(t)$$

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$$\frac{1}{s^2-5s+6} = \frac{1}{s^2-2s-3s+6} = \frac{1}{s(s-2)-3(s-2)} = \frac{1}{(s-3)(s-2)}$$

$$\Rightarrow \frac{1}{(s^2-5s+6)} = \frac{A}{(s-3)} + \frac{B}{(s-2)}$$

$$\Rightarrow 1 = A(s-2) + B(s-3)$$

$s=2 \Rightarrow 0 + B(-1) = 1 \Rightarrow B = -1$

$s=3 \Rightarrow A(1) + 0 = 1 \Rightarrow A = 1$

$$L^{-1}\left[\frac{1}{s^2-5s+6}\right] = L^{-1}\left[\frac{1}{s-3} - \frac{1}{s-2}\right]$$

$$= e^{3t} - e^{2t}$$

Q. $F(s) = \dfrac{s^3}{s^4-a^4}$

$$\frac{s^3}{s^4-a^4} = \frac{s^3}{(s^2-a^2)(s^2+a^2)} = \frac{A}{s^2-a^2} + \frac{B}{s^2+a^2}$$

$$\Rightarrow s^3 = A(s^2+a^2) + B(s^2-a^2)$$

$s^2 = a^2 \Rightarrow sa^2 = A \cdot 2a^2 \Rightarrow A = s/2$

$s^2 = -a^2 \Rightarrow -sa^2 = B \cdot (-2a^2) \Rightarrow B = s/2$

$$L^{-1}\left[\frac{s^3}{s^4-a^4}\right] = L^{-1}\left[\frac{s}{2(s^2-a^2)} + \frac{s}{2(s^2+a^2)}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{s}{s^2-a^2} + \frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{2}[\cosh at + \cos at]$$

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Q. $F(s) = \dfrac{s+4}{s(s-1)(s^2+4)}$

$$\frac{s+4}{s(s-1)(s^2+4)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow s+4 = A(s-1)(s^2+4) + Bs(s^2+4) + (Cs+D)(s)(s-1)$$

$s = 0, s = 1$

$\Rightarrow 4 = -4A \Rightarrow A = -1 \qquad 5 = B \cdot 5 \Rightarrow B = 1$

$$s+4 = A(s^3+4s-s^2-4) + B(s^3+4s) + (Cs+D)(s^2-s)$$

$$s+4 = A(s^3-s^2+4s-4) + B(s^3+4s) + (Cs^3-Cs^2+s^2D-sD)$$

equating co-efficient of s^3:

$$0 = A+B+C \Rightarrow -1+1+C = 0 \Rightarrow C = 0$$

equating co-efficient of s:

$$1 = 4A + 4B - D \Rightarrow -4+4-D = 1 \Rightarrow D = -1$$

$$\mathcal{L}^{-1}\left[\frac{s+4}{s(s-1)(s^2+4)}\right] = \mathcal{L}^{-1}\left[\frac{-1}{s} + \frac{1}{s-1} - \frac{1}{s^2+4}\right]$$

$$= -1 + e^t - \frac{1}{2}\sin 2t$$

Q. $\dfrac{1}{s^2-7s+12} \rightarrow e^{4t} - e^{3t}$

Q. $\dfrac{3s+1}{(s-1)(s^2+1)} \rightarrow e^t - 2\cos t + \sin t$

Q. $\dfrac{2s^2-6s+5}{(s-1)(s-2)(s-3)} \rightarrow \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}$

Q. $\dfrac{1}{(s-2)(s^2+1)} \rightarrow \frac{1}{5}e^{2t} - \frac{1}{5}\cos t - \frac{2}{5}\sin t$

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First shifting theorem

If $L^{-1}F(s) = f(t)$ then $L^{-1}F(s+a) = e^{-at}L^{-1}[F(s)]$

Q. $F(s) = \frac{1}{(s+2)^5}$

$L^{-1}\left[\frac{1}{s^5}\right] = \frac{t^4}{4!}$ $a=2$

$L^{-1}\left[\frac{1}{(s+2)^5}\right] = e^{-2t} \cdot \frac{t^4}{4!} = e^{-2t} L^{-1}\left[\frac{1}{s^5}\right]$

Q. $F(s) = \frac{s}{s^2+4s+13}$ $L^{-1}\left[\frac{s}{s^2+6s+13}\right] = e^{-3t}\cos 2t - 1.5\sin 2t$

$L^{-1}\left[\frac{s}{s^2+4s+13}\right] = L^{-1}\left[\frac{s+2-2}{s^2+4s+13}\right]$

$= L^{-1}\left[\frac{s+2-2}{s^2+4s+4+9}\right]$

$= L^{-1}\left[\frac{s+2-2}{(s+2)^2+3^2}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right]$

$\quad - L^{-1}\left[\frac{2}{(s+2)^2+3^2}\right]$

$= e^{-2t}\cos 3t - \frac{2}{3}L^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$

$= e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t$

Q. $F(s) = \frac{1}{9s^2+6s+1}$

$L^{-1}\left[\frac{1}{9s^2+6s+1}\right] = L^{-1}\left[\frac{1}{(3s+1)^2}\right] = \frac{1}{9}L^{-1}\left[\frac{1}{(s+\frac{1}{3})^2}\right]$

$= \frac{1}{9}e^{-t\cdot\frac{1}{3}}L^{-1}\left[\frac{1}{s^2}\right] = \frac{1}{9}e^{-t/3}t$

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Q. $F(s) = \frac{s+8}{s^2+4s+5}$

$L^{-1}[F(s)] = L^{-1}\left[\frac{s+2+6}{s^2+4s+4+1}\right] = L^{-1}\left[\frac{s+2+6}{(s+2)^2+1}\right]$

$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] + 6L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$

$= e^{-2t} L^{-1}\left[\frac{s}{s^2+1^2}\right] + 6L^{-1}\left[\frac{1}{s^2+1^2}\right]$

$= e^{-2t}[\cos t + 6\sin t]$

Q.

Q. $\frac{s}{(s+3)^2+4} \rightarrow e^{-3t}(\cos 2t - 1.5\sin 2t)$

Q. $\frac{s}{(s+7)^4} \rightarrow e^{-7t}\frac{t^2}{6}(3-7t)$

Q. $\frac{s+2}{s^2-2s-8} \rightarrow e^{t}(\cosh 3t + \sinh 3t)$

Q. $\frac{1}{2(s-1)^2+32} \rightarrow \frac{e^t}{8}\sin 4t \mid \frac{1}{2(s^2+1-2s)+32} = \frac{1}{}$

Q. $\frac{s-4}{4(s-3)^2+16} \rightarrow \frac{1}{4}e^{3t}\cos 2t - \frac{1}{8}e^{3t}\sin 2t$

Q. $\frac{1}{16s^2-32s+16} \rightarrow \frac{1}{16}(e^{t}t$

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Second shifting theorem

$$L^{-1}[e^{-as}F(s)] = f(t-a)u(t-a)$$

Q. $\frac{e^{-\pi s}}{(s+3)} = F(s)$

$$L^{-1}\left[\frac{e^{-\pi s}}{(s+3)}\right] \cdot L^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

$$L^{-1}\left[\frac{e^{-\pi s}}{s+3}\right] = e^{-3(t-\pi)}u(t-\pi) \qquad a = \pi$$

Q. $F(s) = \frac{e^{-s}}{(s+1)^3}$

$$L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2!} \Rightarrow L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t}\frac{t^2}{2!}$$

$$L^{-1}\left[\frac{e^{-s}}{(s+1)^3}\right] = e^{-(t-1)}\frac{(t-1)^2}{2!}u(t-1)$$

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Convolution theorem

If $L[F_1(t)] = f_1(s)$ and $L[F_2(t)] = f_2(s)$ then

$$L\left[\int_0^t f_1(x)\, f_2(t-x)\, dx\right] = F_1(s) \cdot F_2(s)$$

$$\Rightarrow L^{-1}[F_1(s) \cdot F_2(s)] = \int_0^t f_1(x)\, f_2(t-x)\, dx$$

Proof:- L.H.S

$$L\left[\int_0^t f_1(x) f_2(t-x)\, dx\right]$$

~~$$= \int_0^\infty dx \left[\int_x^\infty e^{-st} f_1(x)\, f_2(t-x)\, dt\right]$$~~

$$= \int_0^\infty e^{-st} \left[\int_0^t f_1(x)\, f_2(t-x)\, dx\right] dt$$

$$= \int_0^\infty dx \left[\int_x^\infty e^{-s(t-x+x)} f_1(x)\, f_2(t-x)\, dt\right]$$

$$= \int_0^\infty dx \left[\int_x^\infty e^{-s(t-x)} e^{-sx} f_1(x)\, f_2(t-x)\, dt\right]$$

$$= \int_0^\infty e^{-sx} f_1(x)\, dx \left[\int_x^\infty e^{-s(t-x)} f_2(t-x)\, dt\right]$$

$$= \int_0^\infty e^{-sx} f_1(x)\, dx \int_0^\infty e^{-z} f_2(z)\, dz$$

$t - x = z \Rightarrow dt = dz$

$$= F_1(s)\, F_2(s)$$

Q. Find $L^{-1}\left[\frac{s}{(s^2+4)^2}\right]$

$$\frac{s}{(s^2+4)^2} = \frac{1}{s^2+4} \cdot \frac{s}{s^2+4}$$

$L^{-1}[F_1(s)] = L^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\sin 2t$ $\quad L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t = L^{-1}[F_2(s)]$

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Applying convolution theorem,

$$L^{-1}[F_1(s) \cdot F_2(s)] = \int_0^t f_1(x) f_2(t-x)\, dx$$

$$= \int_0^t \tfrac{1}{2} \sin 2x \cos 2(t-x)\, dx$$

$$= \tfrac{1}{4} \int_0^t [\sin(2x+2t-2x) + \sin(2x-2t+2x)]\, dx$$

$$= \tfrac{1}{4} \int_0^t [\sin 2t + \sin(4x-2t)]\, dx$$

$$= \tfrac{1}{4} \left[x \sin 2t + \frac{\cos(4x-2t)}{4} \right]_0^t$$

$$= \tfrac{1}{4} \left[t \sin 2t - \tfrac{1}{4} \cos(4t-2t) + \tfrac{1}{4} \cos(2t) \right]$$

$$= \tfrac{1}{4} \left[t \sin 2t - \tfrac{1}{4} \cos 2t + \tfrac{1}{4} \cos 2t \right]$$

$$= \tfrac{1}{4} \sin 2t$$

Q. $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

$$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \tfrac{1}{a} \sin at$$

Applying convolution theorem,

$$L^{-1}\left[\frac{1}{s^2+a^2} \cdot \frac{1}{s^2+a^2}\right]$$

$$= \int_0^t \tfrac{1}{a} \sin ax \cdot \tfrac{1}{a} \sin a(t-x)\, dx$$

$$= \tfrac{1}{a^2} \int_0^t \sin ax \sin a(t-x)\, dx$$

$$= \tfrac{1}{a^2} \int_0^t \tfrac{1}{2} [\cos(ax - at + ax) - \cos(ax + at - ax)]\, dx$$

$$= \tfrac{1}{2a^2} \int_0^t [\cos(2ax - at) - \cos at]\, dx$$

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$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin(2ax-at) - x\cos at\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin(2at-at) - t\cos at - \frac{1}{2a}(-\sin at)\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{2a}\sin(at) - t\cos at + \frac{1}{2a}\sin at\right]$$

$$= \frac{1}{2a^2}\left[\frac{1}{a}\sin at - t\cos at\right]$$

$$= \frac{1}{2a^3}[\sin at - at\cos at]$$

Q. $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

$\Rightarrow L^{-1}(1/s) = 1$, $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a}\int_0^t 1\cdot\sin a(t-x)\,dx$$

$$= -\frac{1}{a}\left[\frac{\cos a(t-x)}{-a}\right]_0^t$$

$$= \frac{1}{a^2}[\cos 0 - \cos at]$$

$$= \frac{1}{a^2}(1-\cos at)$$

Q. $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$

$L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2!}$ $\quad L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$$L^{-1}[F_1(s)\cdot F_2(s)] = \int_0^t \sin t\,\frac{1}{2}(t-x)^2\,dx$$

$$= \frac{1}{2}\int_0^t (t^2+x^2-2tx)\sin t\,dx$$

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